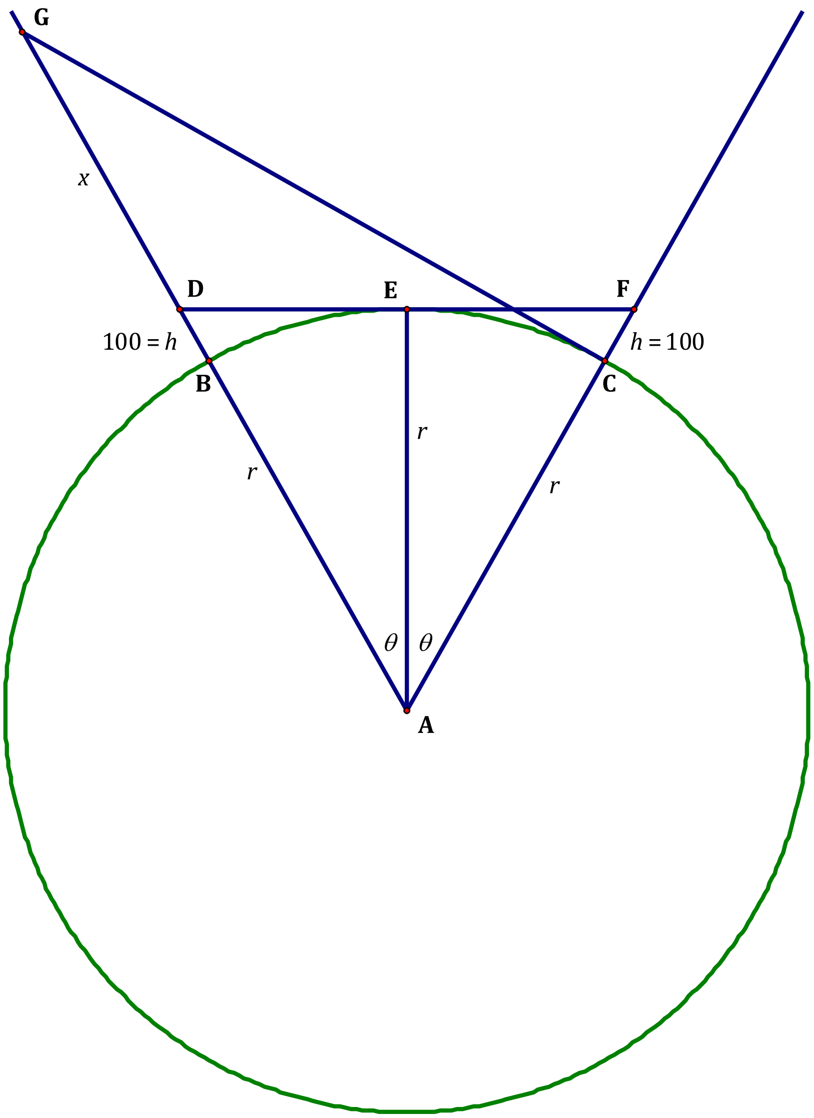
How High Should You Climb Up The Tower?

Riddler Classic of July 1, 2022

As the Royal Astronomer of Planet Xiddler, you and another astronomer set out to investigate the curvature of the planet. The two of you climb two of the tallest towers on the planet, which happen to be in neighboring cities. You both travel 100 meters up each tower on a clear day. Due to the curvature of the planet, you can barely make each other out.

Next, your friend returns to the ground floor of their tower. How high up your tower must you be go so that you can just barely make out your friend again?



Here planet Xiddler is represented by a circle with radius r centered at **A**. The two towers are based at **B** and **C**, and you and your friend have climbed up h = 100 meters to points **D** and **F**. Because of the curvature of the planet, you can just barely see each other, with your line of sight grazing the planet at **E**. When your friend at **F** descends to the ground at **C**, you will need to climb x meters from **D** all the way up to **G** to be able to see him again.

Note that ∠**AEF** and ∠**ACG** are both right angles, allowing us to write two trigonometric equations:

$$\cos\theta = \frac{r}{r+h} \quad \text{and} \quad \cos 2\theta = \frac{r}{r+h+x}$$

Applying one of the double angle formulas for cosine, $\cos 2\theta = 2\cos^2\theta - 1$, gives us:

$$\frac{r}{r+h+x} = 2\left(\frac{r}{r+h}\right)^2 - 1 = \frac{2r^2}{(r+h)^2} - 1 = \frac{2r^2 - (r+h)^2}{(r+h)^2}$$

Inverting the first and last expressions:

$$\frac{r+h+x}{r} = \frac{(r+h)^2}{2r^2 - (r+h)^2}$$

And solving for *x* gives us an expression for how much farther we would need to climb from **D**:

$$\boxed{x = \frac{r(r+h)^2}{2r^2 - (r+h)^2} - (r+h)}$$

As an example, we could use the radius of the Earth for *r*, about 6,371,000 meters, to see how high we would need to climb. Using this value for *r* and 100 meters for *h* gives us:

$$x \approx 300.0157 \text{ meters}$$

Which is very close to the limiting value of 300. In this example the distance between the towers is:

$$d = 2\sqrt{(r+h)^2 - r^2} \approx 71{,}392 \text{ meters} = 71.392 \text{ km}$$

This result suggests that this method *would* work to solve for the radius of planet Xiddler. With some help from Wolfram Alpha, we can solve the boxed equation for *r* in term of *h* and *x*:

$$\boxed{r = h\,\frac{2h + \sqrt{h^2 + 2hx + 2x^2} + x}{x - 3h}}$$