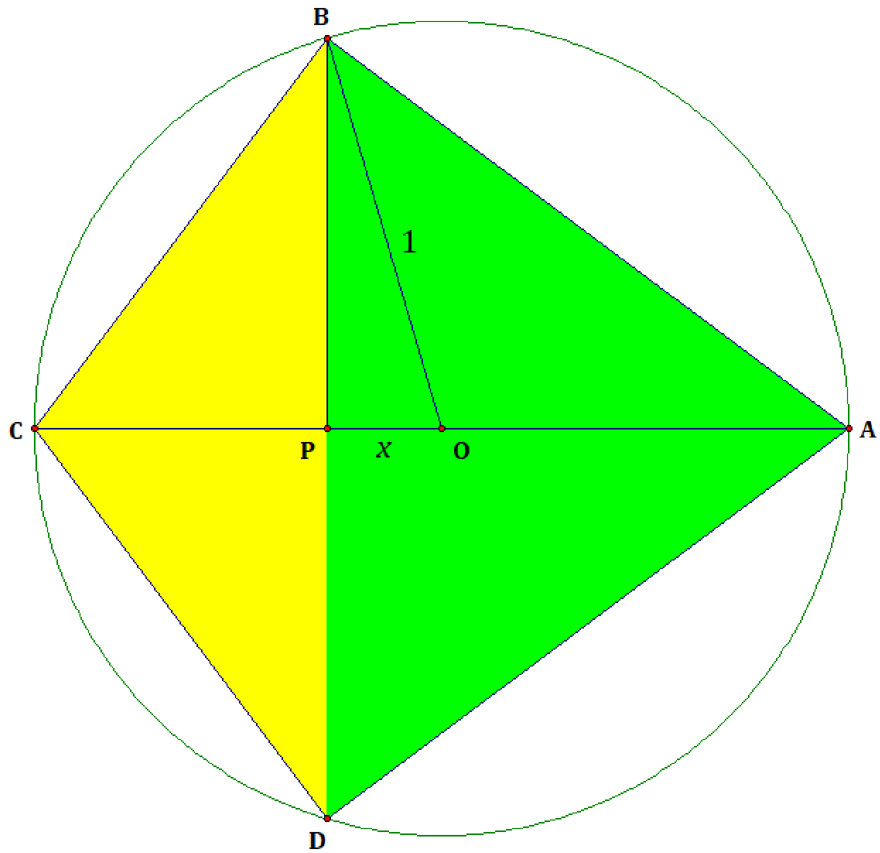
Can You Squeeze Your Quads? Extra Credit Problem

The Fiddler on the Proof of Nov 10, 2023

How can you position four points, **A**, **B**, **C**, & **D**, on a unit circle centered at **O**, to maximize the combined area of the quadrilateral **ABCD**, and a triangle **ABD**?



The symmetries of the problem convince me that **AC** must be a diameter, and that **BD** must be perpendicular to **AC**. That means that the only significant variable is the distance between point **O**, the center of the circle, and point **P,** the intersection of **AC** and **BD**. We shall call that distance: *x***.**

$$\mathbf{PB} = \sqrt{1 - x^2}$$

$$\text{Area of } \mathbf{ABCD} = 2\sqrt{1 - x^2}$$

$$\text{Area of } \mathbf{ABD} = (x + 1)\sqrt{1 - x^2}$$

$$\text{Total Area} = 2\sqrt{1 - x^2} + (x + 1)\sqrt{1 - x^2} = \boxed{(x + 3)\sqrt{1 - x^2}}$$

To maximize this function with respect to x, I relied on my friend, Wolfram Alpha:

maximize (x+3)*sqrt(1-x^2)

Input interpretation

maximize $(x+3)\sqrt{1-x^2}$

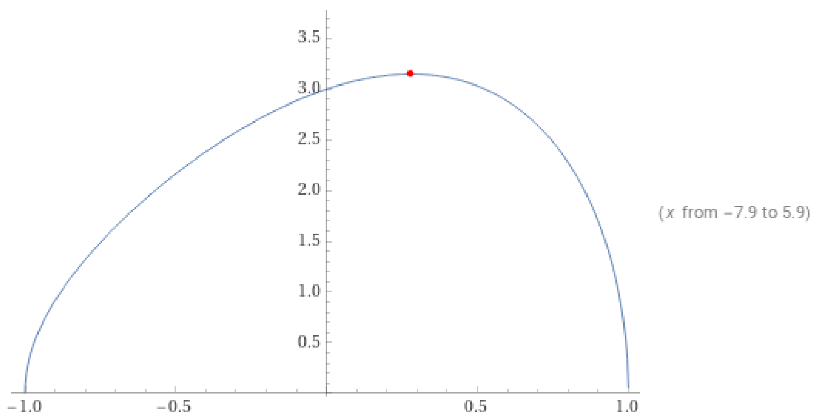
Global maximum Exact form

$$\max\{(x+3)\sqrt{1-x^2}\} = \frac{1}{8}\sqrt{214+102\sqrt{17}} \text{ at } x = \frac{\sqrt{17}}{4} - \frac{3}{4}$$

Global maximum Approximate form

$$\max\{(x+3)\sqrt{1-x^2}\} \approx 3.1488 \text{ at } x \approx 0.28078$$

Plot



Therefore the maximum combined area of the quadrilateral and triangle is:

$$\frac{1}{8}\sqrt{214 + 102\sqrt{17}} \approx 3.1488$$

Which is delightfully close to **π**, the area of the circumscribed unit circle!

Dean Ballard
Nov. 12, 2023