Can You Solve March Madness?

Riddler Classic – March 26, 2021

What is the greatest number of distinct shooting profiles that are made up of three *different* probabilities — *p*, *q* and *r*, in some order for the three shots — that can result in the same overall expected number of points?

First, we calculate the expected value assuming probabilities *p*, *q*, and *r* are for the 1st, 2nd, and 3rd shots respectively. If we miss the 1st shot, we get zero points: $P(0) = 1 - p$. If we make 1st shot, but miss the 2nd, we get one point: $P(1) = p(1 - q)$. If we make the first two shots, but miss the 3rd, we get two points: $P(2) = pq(1 - r)$. And finally, if we make all three shots, we get three points: $P(3) = pqr$. The expected value is just the sum of each of the point values times their probability.

$$EV = 0 \cdot (1 - p) + 1 \cdot p(1 - q) + 2 \cdot pq(1 - r) + 3 \cdot pqr$$

$$EV = p - pq + 2pq - 2pqr + 3pqr$$

$$\boldsymbol{EV = p + pq + pqr}$$

The question is, can we get the *same* expected value if we reassign our probabilities *p*, *q*, and *r* to different shot orders. There are six possible permutations of *p*, *q*, and *r*, listed here: {***pqr*, *qpr*, *rqp*, *prq*, *rpq*, *qrp***}. For example, the ***qpr*** case would mean that the probability of the 1st shot being made equals *q*, the probability of the 2nd shot being made equals *p*, and the probability of the 3rd shot being made equals *r*. That gives us five rearrangements of the initial ordering to consider:

1. *qpr*: This means the expected value might stay the same if we interchange *p* and *q*:

$$p + pq + pqr = q + qp + qpr$$

$$p = q$$

But the problem statement requires that *p*, *q*, and *r* be different, so ***qpr*** fails.

2. *rqp*: This means the expected value might stay the same if we interchange *p* and *r*:

$$p + pq + pqr = r + rq + rqp$$

$$p(1 + q) = r(1 + q)$$

$$p = r$$

Again, the problem statement requires that *p*, *q*, and *r* be different, so ***rqp*** fails.

3. *prq*: This means the expected value might stay the same if we interchange *q* and *r*:

$$p + pq + pqr = p + pr + prq$$

$$pq = pr$$

$$q = r$$

Once again, the problem statement requires that *p*, *q*, and *r* be different, so ***prq*** fails.

4. ***rpq***: In this case we substitute p with r, q with p, and r with q:

$$p + pq + pqr = r + rp + rpq$$

$$p + pq = r + rp = r(1 + p)$$

$$r = \frac{p + pq}{(1 + p)}, \;\; p \neq q$$

This looks promising. We just need to make sure that $r \in [0, 1]$

Set $r \geq 0$

$$\frac{p + pq}{(1 + p)} \geq 0, \qquad p + pq \geq 0$$

which is always true for $p, q \in [0, 1]$

Set $r \leq 1$

$$\frac{p + pq}{(1 + p)} \leq 1, \qquad p + pq \leq 1 + p, \qquad pq \leq 1$$

which is always true for $p, q \in [0, 1]$

So, this value of r will work for any $p, q \in [0, 1]$

Now we just need to make sure that our choice of r doesn't end up equal to p or q. Check:

$$r = p = \frac{p + pq}{(1 + p)}$$

$$p + p^2 = p + pq, \qquad p^2 = pq, \qquad p = q$$

We have already eliminated $p = q$, so no problem.

$$r = q = \frac{p + pq}{(1 + p)}$$

$$q + pq = p + pq, \qquad p = q$$

Again, we have already eliminated $p = q$, so no problem.

$$\boxed{\text{To summarize } \boldsymbol{rpq}\text{, we can choose:} \quad p \in [0, 1], \qquad q \in [0, 1], q \neq p, \qquad r = \frac{p + pq}{(1 + p)}}$$

Here's an example:

$$p = 0.5, \qquad q = 0.8$$

$$r = \frac{p + pq}{(1 + p)} = \frac{0.5 + 0.5 \cdot 0.8}{(1 + 0.5)} = \frac{0.5 + 0.4}{1.5} = \frac{0.9}{1.5} = 0.6$$

We check to see if the expected values are the same:

$$p + pq + pqr = r + rp + rpq$$

$$0.5 + 0.5 \cdot 0.8 + 0.5 \cdot 0.8 \cdot 0.6 = 0.6 + 0.6 \cdot 0.5 + 0.6 \cdot 0.5 \cdot 0.8$$

$$0.5 + 0.4 + 0.24 = 0.6 + 0.3 + 0.24$$

$$1.14 = 1.14$$

5. ***qrp***: For our final case we substitute p with q, q with r, and r with p:

$$p + pq + pqr = q + qr + qrp$$
$$p + pq = q + qr$$
$$qr = p - q + pq$$
$$r = \frac{p + pq - q}{q}, \quad p \neq q, q \neq 0$$

This also looks promising. We just need to make sure that $r \in [0, 1]$

$$\text{Set } \ r \geq 0$$
$$\frac{p + pq - q}{q} \geq 0, \qquad p + pq - q \geq 0$$
$$q - pq - p \leq 0, \qquad q(1 - p) \leq p$$
$$q \leq \frac{p}{1 - p}$$

So, if q is chosen to be less than $\frac{p}{1-p}$, then r will greater than or equal to 0.

$$\text{Set } \ r \leq 1$$
$$\frac{p + pq - q}{q} \leq 1, \qquad p + pq - q \leq q$$
$$pq - 2q \leq -p, \qquad 2q - pq \geq p, \qquad q(2 - p) \geq p$$
$$q \geq \frac{p}{2 - p}$$

And, if q is chosen to be greater than $\frac{p}{2-p}$, then r will be less than or equal to 1.

So, this value of r will work for any $p \in [0, 1]$ and any $q \in \left[\frac{p}{2-p}, \frac{p}{1-p}\right], \ q \neq p$.

Now we just need to make sure that our choice of r doesn't end up equal to p or q. Check:

$$r = p = \frac{p + pq - q}{q}, \quad q \neq 0$$
$$pq = \ p + pq - q, \qquad 0 = \ p - q, \qquad p = q$$

We have already eliminated $p = q$, so no problem.

$$r = q = \frac{p + pq - q}{q}, \quad q \neq 0$$
$$q^2 = p + q(p - 1), \qquad q^2 + (1 - p)q + p = 0$$

For $p \in [0, 1]$ and $q \in (0, 1]$, each term of this polynomial is greater than or equal to zero, so again no problem.

$$\boxed{\text{To summarize } \boldsymbol{qrp}\text{, we choose:} \quad p \in [0, 1], \quad q \in \left[\frac{p}{2 - p}, \frac{p}{1 - p}\right], q \neq p, \quad r = \frac{p + pq - q}{q}}$$

Here's an example:

$$p = 0.4, \qquad \frac{0.4}{2-0.4} \leq q \leq \frac{0.4}{1-0.4}, \qquad \frac{1}{4} \leq q \leq \frac{2}{3}, \qquad q = 0.5$$

$$r = \frac{p + pq - q}{q} = \frac{0.4 + 0.4 \cdot 0.5 - 0.5}{0.5} = \frac{0.1}{0.5} = 0.2$$

We check to see if the expected values are the same:

$$p + pq + pqr = q + qr + qrp$$

$$0.4 + 0.4 \cdot 0.5 + 0.4 \cdot 0.5 \cdot 0.2 = 0.5 + 0.5 \cdot 0.2 + 0.5 \cdot 0.2 \cdot 0.4$$

$$0.4 + 0.2 + 0.04 = 0.5 + 0.1 + 0.04$$

$$0.64 = 0.64$$

Now it would be nice if the same values of p, q, and r, would work for *both* the ***rpq*** case and the ***qrp*** case. For that we would need to find values of p and q that give the *same* value of r in both equations. That is:

$$r = \frac{p + pq}{(1+p)} = \frac{p + pq - q}{q}$$

$$(p + pq)q = (1+p)(p + pq - q)$$

$$pq + pq^2 = p + pq - q + p^2 + p^2q - pq$$

$$pq^2 - p^2q + pq - p^2 - p + q = 0$$

There are two solutions to this equation (thank you, Wolfram Alpha).

Solution #1:

$$p = q$$

is ruled out by the requirement that $p \neq q$.

And solution #2:

$$q = -\frac{p+1}{p}, \;\; p \neq 0$$

gives only negative values for q when $p \in (0, 1]$, so that also fails.

In summary, certain values of p, q, and r can produce the same expected value for an RCAA triple free throw for different assignments of probabilities to free throw order. We have **two cases** where this can occur, one case pairing orders ***pqr*** and ***rpq***, and the other case pairing orders ***pqr*** and ***qrp***.

Dean Ballard
March 28, 2021