Can You Solve March Madness?
Riddler Classic – March 26, 2021

What is the greatest number of distinct shooting profiles that are made up of three different probabilities — $p$, $q$ and $r$, in some order for the three shots — that can result in the same overall expected number of points?

First, we calculate the expected value assuming probabilities $p$, $q$, and $r$ are for the 1st, 2nd, and 3rd shot respectively. If we miss the 1st shot, we get zero points: $P(0) = 1 - p$. If we make 1st shot, but miss the 2nd, we get one point: $P(1) = p(1 - q)$. If we make the first two shots, but miss the 3rd, we get two points: $P(2) = pq(1 - r)$. And finally, if we make all three shots, we get three points: $P(3) = pqr$. The expected value is just the sum of each of the point values times their probability.

$$EV = 0 \cdot (1 - p) + 1 \cdot p(1 - q) + 2 \cdot pq(1 - r) + 3 \cdot pqr$$

The question is, can we get the same expected value if we reassign our probabilities $p$, $q$, and $r$ to different shot orders. There are six possible permutations of $p$, $q$, and $r$, listed here: \{pqr, qpr, rqp, prq, rpq, qrp\}. For example, the qpr case would mean that the probability of the 1st shot being made equals $q$, the probability of the 2nd shot being made equals $p$, and the probability of the 3rd shot being made equals $r$. That gives us five rearrangements of the initial ordering to consider:

1. **qpr**: This means the expected value might stay the same if we interchange $p$ and $q$:
   $$p + pq + pqr = q + qp + qpr$$
   $$p = q$$
   But the problem statement requires that $p$, $q$, and $r$ be different, so qpr fails.

2. **rqp**: This means the expected value might stay the same if we interchange $p$ and $r$:
   $$p + pq + pqr = r + rq + rqp$$
   $$p(1 + q) = r(1 + q)$$
   $$p = r$$
   Again, the problem statement requires that $p$, $q$, and $r$ be different, so rqp fails.

3. **prq**: This means the expected value might stay the same if we interchange $q$ and $r$:
   $$p + pq + pqr = p + pr + prq$$
   $$pq = pr$$
   $$q = r$$
   Once again, the problem statement requires that $p$, $q$, and $r$ be different, so prq fails.
4. \( rpq \): In this case we substitute \( p \) with \( r \), \( q \) with \( p \), and \( r \) with \( q \):

\[
p + pq + pqr = r + rp + rqp \\
p + pq = r + rp = r(1 + p) \\
r = \frac{p + pq}{(1 + p)}, \quad p \neq q
\]

This looks promising. We just need to make sure that \( r \in [0, 1] \)

Set \( r \geq 0 \)

\[
\frac{p + pq}{(1 + p)} \geq 0, \quad p + pq \geq 0
\]

which is always true for \( p, q \in [0, 1] \)

Set \( r \leq 1 \)

\[
\frac{p + pq}{(1 + p)} \leq 1, \quad p + pq \leq 1 + p, \quad pq \leq 1
\]

which is always true for \( p, q \in [0, 1] \)

So, this value of \( r \) will work for any \( p, q \in [0, 1] \)

Now we just need to make sure that our choice of \( r \) doesn’t end up equal to \( p \) or \( q \). Check:

\[
r = p = \frac{p + pq}{(1 + p)} \\
p + p^2 = p + pq, \quad p^2 = pq, \quad p = q
\]

We have already eliminated \( p = q \), so no problem.

\[
r = q = \frac{p + pq}{(1 + p)} \\
q + pq = p + pq, \quad p = q
\]

Again, we have already eliminated \( p = q \), so no problem.

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To summarize \( rpq \), we can choose: \( p \in [0, 1], \quad q \in [0, 1], q \neq p, \quad r = \frac{p + pq}{(1 + p)} \)

Here’s an example:

\[
p = 0.5, \quad q = 0.8 \\
r = \frac{p + pq}{(1 + p)} = \frac{0.5 + 0.5 \cdot 0.8}{(1 + 0.5)} = \frac{0.5 + 0.4}{1.5} = \frac{0.9}{1.5} = 0.6
\]

We check to see if the expected values are the same:

\[
p + pq + pqr = r + rp + rqp \\
0.5 + 0.5 \cdot 0.8 + 0.5 \cdot 0.8 \cdot 0.6 = 0.6 + 0.6 \cdot 0.5 + 0.6 \cdot 0.5 \cdot 0.8 \\
0.5 + 0.4 + 0.24 = 0.6 + 0.3 + 0.24 \\
1.14 = 1.14
5. \textbf{qrp}: For our final case we substitute \( p \) with \( q \), \( q \) with \( r \), and \( r \) with \( p \):

\[
\begin{align*}
    p + pq + pqr &= q + qr + qrp \\
    p + pq &= q + qr \\
    qr &= p - q + pq \\
    r &= \frac{p + pq - q}{q}, \quad p \neq q, q \neq 0
\end{align*}
\]

This also looks promising. We just need to make sure that \( r \in [0,1] \)

Set \( r \geq 0 \)

\[
\begin{align*}
    \frac{p + pq - q}{q} &\geq 0, \quad p + pq - q \geq 0 \\
    q - pq - p &\leq 0, \quad q(1 - p) \leq p \\
    q &\leq \frac{p}{1 - p}
\end{align*}
\]

So, if \( q \) is chosen to be less than \( \frac{p}{1-p} \), then \( r \) will be greater than or equal to 0.

Set \( r \leq 1 \)

\[
\begin{align*}
    \frac{p + pq - q}{q} &\leq 1, \quad p + pq - q \leq q \\
    pq - 2q &\leq -p, \quad 2q - pq \geq p, \quad q(2 - p) \geq p \\
    q &\geq \frac{p}{2 - p}
\end{align*}
\]

And, if \( q \) is chosen to be greater than \( \frac{p}{2-p} \), then \( r \) will be less than or equal to 1.

So, this value of \( r \) will work for any \( p \in [0,1] \) and any \( q \in \left[ \frac{p}{2-p}, \frac{p}{1-p} \right], \; q \neq p \).

Now we just need to make sure that our choice of \( r \) doesn't end up equal to \( p \) or \( q \). Check:

\[
\begin{align*}
    r &= p = \frac{p + pq - q}{q}, \quad q \neq 0 \\
    pq &= p + pq - q, \quad 0 = p - q, \quad p = q \\
\end{align*}
\]

We have already eliminated \( p = q \), so no problem.

\[
\begin{align*}
    r &= q = \frac{p + pq - q}{q}, \quad q \neq 0 \\
    q^2 &= p + q(p - 1), \quad q^2 + (1 - p)q + p = 0
\end{align*}
\]

For \( p \in [0,1] \) and \( q \in (0,1) \), each term of this polynomial is greater than or equal to zero, so again no problem.

\[
\text{To summarize \textbf{qrp}, we choose: } \quad p \in [0,1], \quad q \in \left[ \frac{p}{2-p}, \frac{p}{1-p} \right], q \neq p, \quad r = \frac{p + pq - q}{q}
\]
Here's an example:

\[ p = 0.4, \quad \frac{0.4}{2 - 0.4} \leq q \leq \frac{0.4}{1 - 0.4}, \quad \frac{1}{4} \leq q \leq \frac{2}{3}, \quad q = 0.5 \]

\[ r = \frac{p + pq - q}{q} = \frac{0.4 + 0.4 \cdot 0.5 - 0.5}{0.5} = \frac{0.1}{0.5} = 0.2 \]

We check to see if the expected values are the same:

\[ p + pq + pqr = q + qr + qrp \]

\[ 0.4 + 0.4 \cdot 0.5 + 0.4 \cdot 0.5 \cdot 0.2 = 0.5 + 0.5 \cdot 0.2 + 0.5 \cdot 0.2 \cdot 0.4 \]

\[ 0.4 + 0.2 + 0.04 = 0.5 + 0.1 + 0.04 \]

\[ 0.64 = 0.64 \]

Now it would be nice if the same values of \( p, q, \) and \( r \), would work for both the \( rpq \) case and the \( qrp \) case. For that we would need to find values of \( p \) and \( q \) that give the same value of \( r \) in both equations. That is:

\[ r = \frac{p + pq}{1 + p} = \frac{p + pq - q}{q} \]

\[ (p + pq)q = (1 + p)(p + pq - q) \]

\[ pq + pq^2 = p + pq - q + p^2 + p^2q - pq \]

\[ pq^2 - p^2q + pq - p^2 - p + q = 0 \]

There are two solutions to this equation (thank you, Wolfram Alpha).

Solution #1:

\[ p = q \]

is ruled out by the requirement that \( p \neq q \).

And solution #2:

\[ q = -\frac{p + 1}{p}, \quad p \neq 0 \]

gives only negative values for \( q \) when \( p \in (0, 1] \), so that also fails.

In summary, certain values of \( p, q, \) and \( r \) can produce the same expected value for an RCAA triple free throw for different assignments of probabilities to free throw order. We have two cases where this can occur, one case pairing orders \( pqr \) and \( rpq \), and the other case pairing orders \( pqr \) and \( qrp \).

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