The Unfair Coin

Flipping a coin is a traditional and effective way to make a "fair" choice between two alternatives. We flip a coin at the start of a football game to decide which team gets to kick off, and which team gets to receive. The presumption is that the coin is fair – that the probability of landing on heads is exactly equal to the probability of landing on tails. This implies that both teams will have the same chance of winning the coin toss. Mathematically we say: $p(H) = p(T) = 0.5$

If instead we flipped an "unfair coin", one where $p(H) \neq 0.5$, then the resulting choice of team would also be unfair, and no one wants that to happen. However, it may be possible to make a *fair* choice between two alternatives with an *unfair* coin if we can flip the coin more than once!

Imagine a coin where the probability of landing on heads is equal to $\sqrt{1/2}$. This would result in the following probabilities for heads and tails, exactly and as approximate decimals:

$$p(H) = \sqrt{1/2} = \frac{\sqrt{2}}{2} \approx 0.7071, \qquad p(T) = 1 - p(H) = \frac{2}{2} - \frac{\sqrt{2}}{2} = \frac{2-\sqrt{2}}{2} \approx 0.2929$$

Flipping a coin twice gives us four possible outcomes: HH, HT, TH, and TT. For an unfair coin with $p(H) = \sqrt{1/2}$, the probabilities for each of these outcomes are:

$$p(HH) = \left(\sqrt{1/2}\right)^2 = 0.5, \qquad p(HT) = p(TH) = \frac{\sqrt{2}}{2} \cdot \frac{2-\sqrt{2}}{2} = \frac{\sqrt{2}-1}{2}, \qquad p(TT) = \left(\frac{2-\sqrt{2}}{2}\right)^2 = \frac{3}{2} - \sqrt{2}$$

This means that we can make a fair choice between team A and team B by letting team A "win the toss" whenever heads comes up twice, otherwise team B wins. The probability of team B winning would therefore be the sum: $p(HT) + p(TH) + p(TT)$.

$$= 2\left(\frac{\sqrt{2}-1}{2}\right) + \left(\frac{2-\sqrt{2}}{2}\right)^2 = \left(\sqrt{2}-1\right) + \left(\frac{3}{2} - \sqrt{2}\right) = \sqrt{2} - \sqrt{2} + \frac{3}{2} - 1 = 0.5$$

Of course, this simply confirms the fact that:

$$p(HT) + p(TH) + p(TT) = 1 - p(HH) = 1 - 0.5 = 0.5$$

It turns out that the *only* unfair coin that can be used to make a fair choice with two flips is one that has a probability of heads (or tails) of $\sqrt{1/2}$.

However, if we allow ourselves to make *three* flips of the coin, then other possibilities arise. Can you find all of the unfair coins that could be used to make a fair choice with three coin flips?