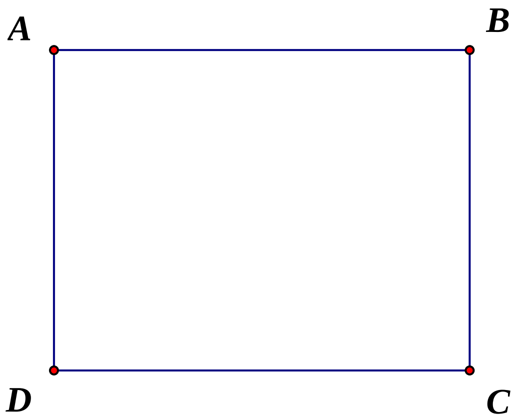
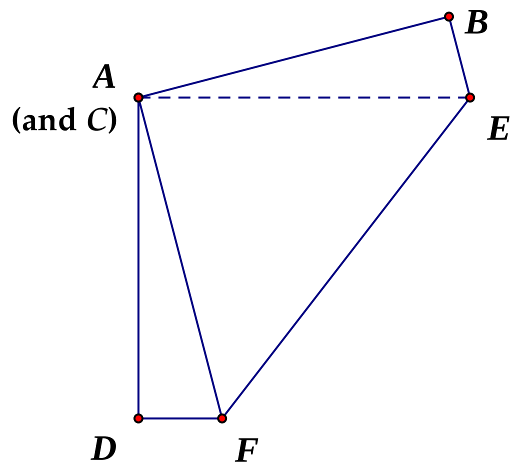
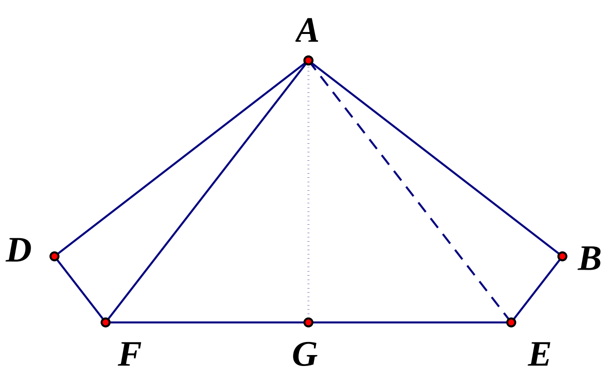
Folding a Pentagon

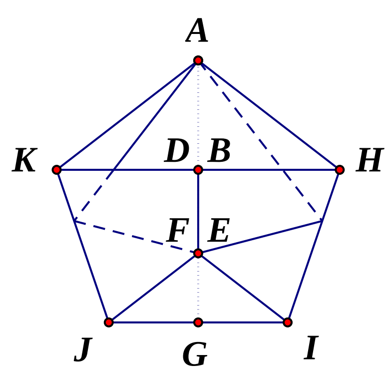
Here's an interesting way to fold a piece of paper into a pentagon. Start with a standard sized piece of paper (8½" by 11"). Fold corner C over to corner A, making a crease we'll call EF.





Turn segment EF to the bottom, and fold the whole thing in half, bringing corner B to corner D. Then unfold, leaving a crease from A to G. Finally, fold both edges DF and BE to exactly meet each other at the central crease from A to G.

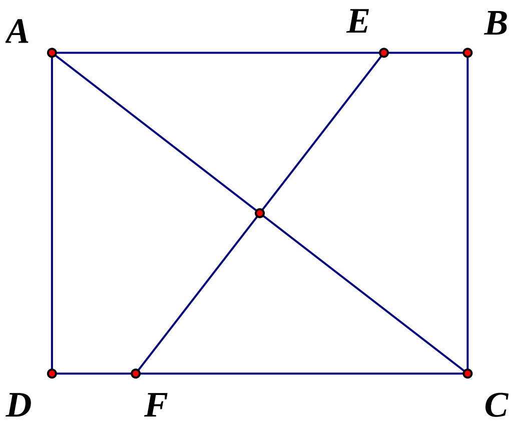


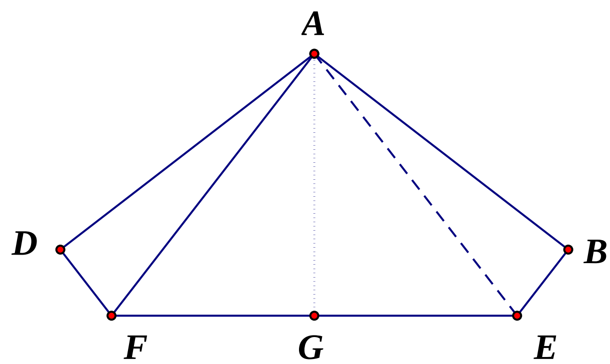


Now, you have made a pentagon AHIJK. It is *almost*, but not quite, a regular pentagon.

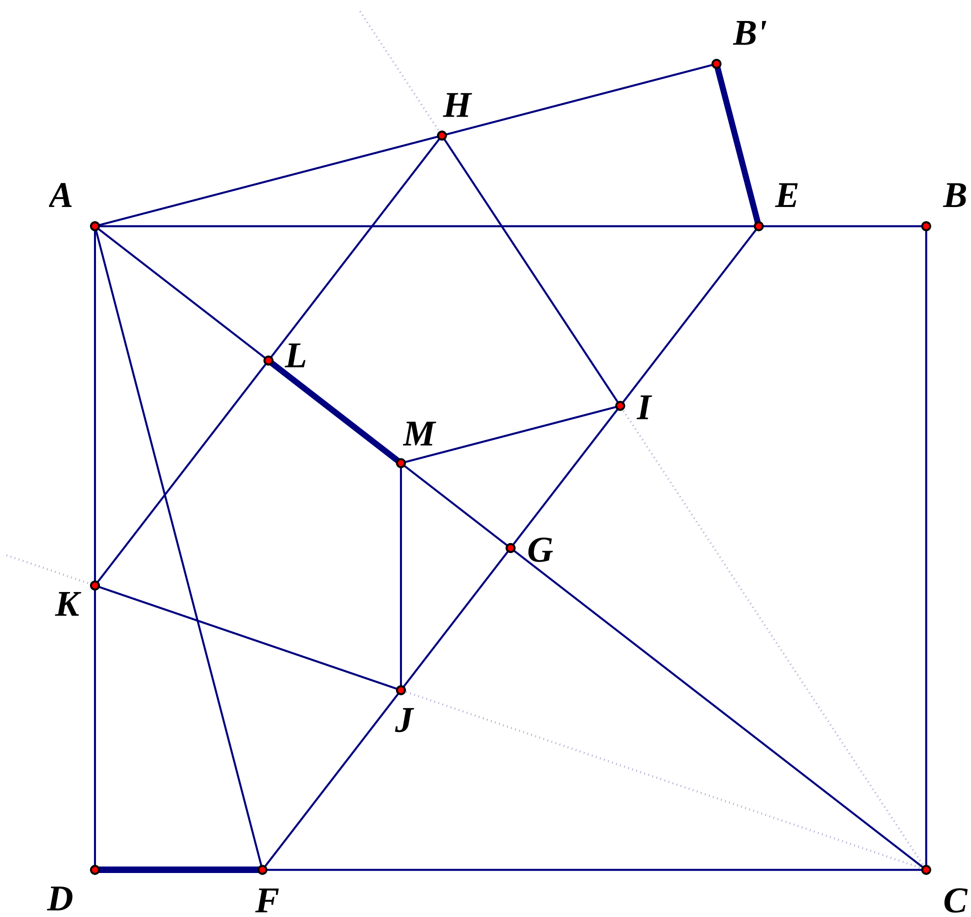
Question: Are there rectangular dimensions which would result in a *regular* pentagon after this folding, and if so, what are those dimensions?

To answer this question, let us do a little geometry. We start by noting that the crease of the first fold must be along the perpendicular bisector of the diagonal AC. We also see that after we turn and fold it in half, the crease from A to G is along the same line as the diagonal AC.



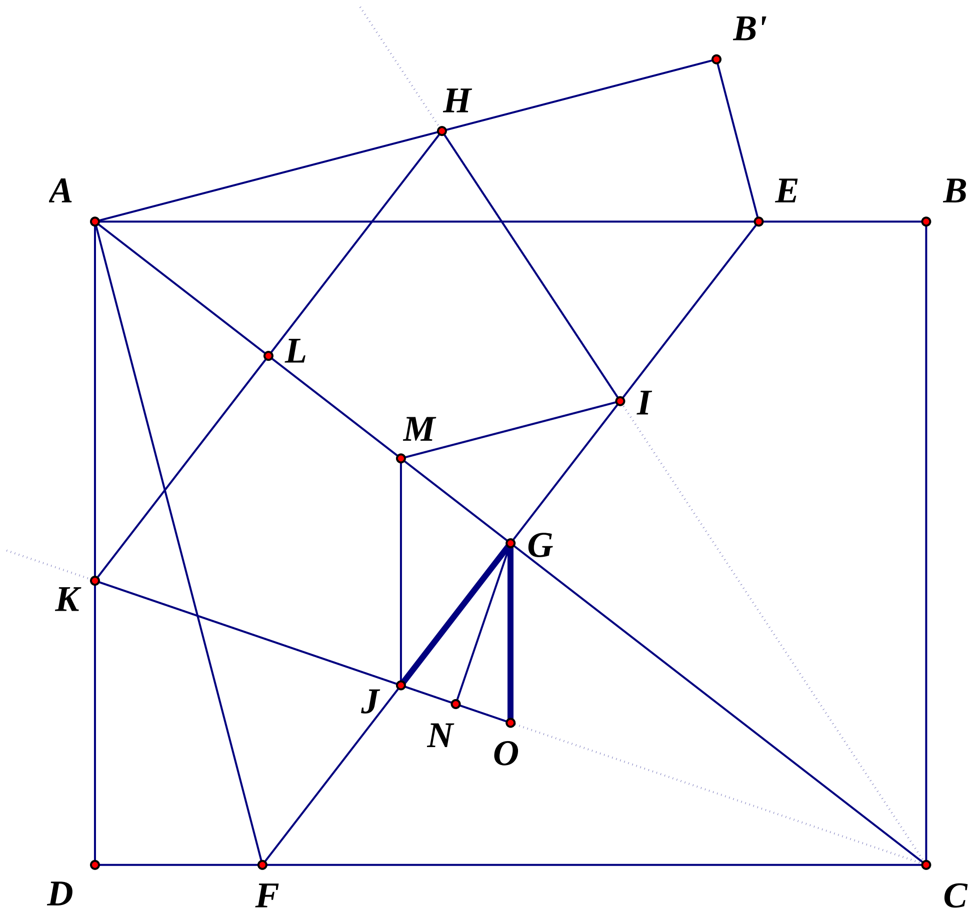


The final fold, bringing segments EB and DF to the center line AG must be along the angle bisectors of ∠ACE and ∠ACF. Here is what those fold lines look like when the paper is unfolded and returned to its original orientation:



Here segment DF and B'E are unfolded from LM, segment BC is unfolded from B'A, and the pentagon is identified as AHIJK.

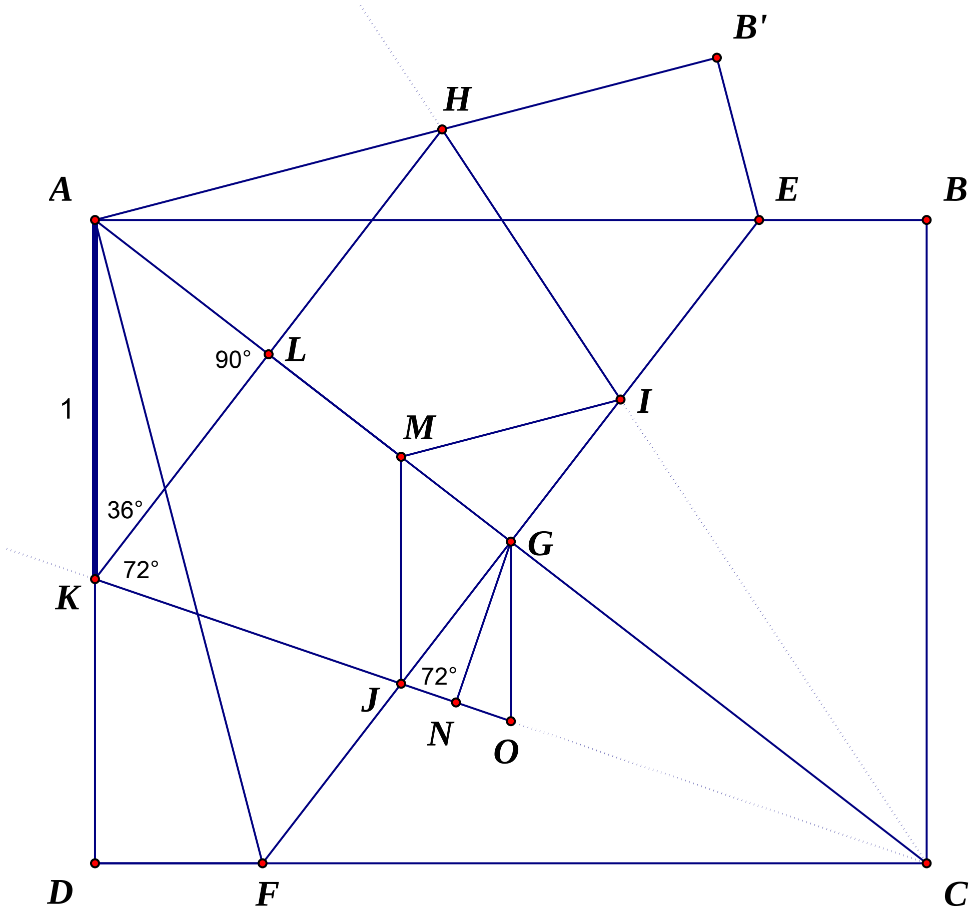
We start by constructing a point N on the ray CK such that GN ⊥ CK. We also construct a point O on the ray CK such that GO ⊥ DC.



Since GJ, GN, and GO are perpendicular respectively to AC, KC, and DC, then GN must be the angle bisector of ∠JGO, just as CK is the angle bisector of ∠ACD. Therefore, by Angle-Side-Angle, ΔJGN ≅ ΔOGN, and in particular *m*GJ = *m*GO. Since G is the midpoint of AC, and GO||AK, GO is a mid-segment of ΔACK, making *m*GO one half *m*AK. By symmetry *m*GJ = *m*GI, and *m*AK = *m*AH, so we now know that three sides of the pentagon, AK, AH, and JI, must always be the same length, independent of the dimensions of the rectangle.

A regular pentagon has interior angles of 108°, so we start by setting *m*∠AKJ = 108°. That means *m*∠DKC = 72°, so its complement *m*∠KCD = 18°. Remembering that KC is the angle bisector of ∠ACD, give us *m*∠ACD = 36°, and its complement *m*∠DAC = 54°. By symmetry, *m*∠KAH is twice *m*∠DAC, so *m*∠KAH = 108°. Symmetry again gives us *m*∠AHI = *m*∠AKJ = 108°. ∠KJI is an exterior angle of ΔJGC, so *m*∠KJI = *m*∠JGC + *m*∠CGJ = 90° + 18° = 108°, and *m*∠KJI = *m*∠HIJ by symmetry. Thus, when *m*∠AKJ = 108° all five pentagon angles will measure 108°.

Given these angles, we note that ΔAKH is isosceles with $m\angle KAH = 108°$, so $m\angle AKH = 36°$ and $m\angle ALK = 90°$. If we assign mAK = 1, then the mKL = cos(36°). Now $m\angle LKC = 108° - 36° = 72°$, so mKC = mKL / cos(72°). Similarly, $m\angle GJC = 72°$, so mJC = mGJ / cos(72°).



This lets us calculate mKJ = mKC – mJC = (mKL – mGJ)/ cos(72°). Using exact values for cos(36°) = $(\sqrt{5}+1)/4$ and cos(72°) = $(\sqrt{5}-1)/4$, we get:

$$m\text{KJ} = \frac{m\text{KX} - m\text{GJ}}{\cos(72°)} = \frac{\frac{\sqrt{5}+1}{4} - \frac{1}{2}}{\frac{\sqrt{5}-1}{4}} = \frac{\frac{(\sqrt{5}+1)-2}{4}}{\frac{\sqrt{5}-1}{4}} = \frac{\frac{\sqrt{5}-1}{4}}{\frac{\sqrt{5}-1}{4}} = 1$$

Thus mKJ = mAK, and by symmetry, mHI = mAK. This demonstrates that all five sides are equal in length when all five angles measure 108°.

Therefore, if we start with a rectangle whose ratio of long side to short side is tan(∠DAC = 54°) (≈ 1.37638), then this folding *will* give us a regular pentagon. It's convenient that this number is very, very close to 1⅜, which is a ratio of 11 to 8. So, if you just trim ½" off the long edge of your original 8½" by 11"paper, you can make a nearly perfect regular pentagon. Try it!